

# 2-SELMER GROUPS AND THE BIRCH-SWINNERTON-DYER CONJECTURE FOR THE CONGRUENT NUMBER CURVE

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**ABSTRACT.** We take an approach toward counting the number of  $n$  for which the curves  $E_n : y^2 = x^3 - n^2x$  have 2-Selmer groups of a given size. This question was also discussed in a pair of papers by Roger Heath-Brown [6, 7]. We discuss the connection between computing the size of these Selmer groups and verifying cases of the Birch and Swinnerton-Dyer Conjecture. The key ingredient for the asymptotic formulae is the “independence” of the Legendre symbol evaluated at the prime divisors of an integer with exactly  $k$  prime factors.

## 1. INTRODUCTION

A problem dating back to the tenth century, is to determine which positive integers  $n$  are the areas of right triangles with rational side lengths. Such integers are called *congruent numbers*. The problem of determining whether or not  $n$  is a congruent number is related to computing the rank of the elliptic curve

$$(1.1) \quad E_n : y^2 = x^3 - n^2x.$$

It is well known [10] that a positive square-free integer  $n$  is congruent if and only if the rank of  $E_n(\mathbb{Q})$ , say  $r(n)$ , is non-zero. This criterion has led to many infinite families of congruent numbers. For example, see [4], if  $p$  and  $q$  are distinct primes, then we have:

- Heegner:  $2p$  is a congruent number when  $p \equiv 3 \pmod{8}$
- Monsky:  $2pq$  is congruent whenever  $p \equiv 1 \pmod{8}$ ,  $q \equiv 3, 7 \pmod{8}$  and  $\left(\frac{p}{q}\right) = -1$ .

Similarly, there are many results that yield infinite families of non-congruent numbers. For example, if  $p, q, r, p_j$  are distinct primes, then we have:

- Lagrange:  $pqr$  is non-congruent when  $p, q \equiv 1 \pmod{8}$ ,  $r \equiv 3 \pmod{8}$ , and  $\left(\frac{p}{q}\right) = \left(\frac{p}{r}\right) = -1$ , [4],
- Iskra:  $p_1 \cdots p_\ell$  is non-congruent when  $p_j \equiv 3 \pmod{8}$  for all  $j$  and  $\left(\frac{p_j}{p_k}\right) = -1$  for all  $j < k$ , [8].

These examples show how the rank of  $E_n$  is intrinsically related to the quadratic relationships between the prime factors of  $n$ . We will see this in our analysis as well.

We are interested in the size of three different Selmer groups, which we refer to as the 2-Selmer groups. Let  $[2]$  be the multiplication by 2 map, and let  $\phi$  and  $\hat{\phi}$  denote the degree 2 isogenies such that  $\phi\hat{\phi} = [2]$ . More precisely, the 2-dual curve of  $E_n$  is  $E'_n : y^2 = x^3 + 4n^2x$

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and  $\phi : E_n \rightarrow E'_n$  is defined by  $\phi(x, y) = (y^2/x^2, y(n^2 - x^2)/x^2)$ . The 2-Selmer groups are the Selmer groups associated to these maps, namely,  $S^{(2)}(E_n)$ ,  $S^{(\phi)}(E_n)$ , and  $S^{(\hat{\phi})}(E_n)$ . It is a fact that  $|S^{(2)}(E_n)| = 2^{2+s(n)}$ ,  $|S^{(\phi)}(E_n)| = 2^{s^\phi(n)}$ , and  $|S^{(\hat{\phi})}(E_n)| = 2^{2+s^{\hat{\phi}}(n)}$  for non-negative integers  $s(n)$ ,  $s^\phi(n)$ , and  $s^{\hat{\phi}}(n)$ . The fundamental inequality that relates  $s(n)$  and  $r(n)$  is

$$(1.2) \quad r(n) \leq s(n).$$

Thus information about  $s(n)$  allows us to gather information about  $r(n)$ . The method of computing  $S^{(2)}(E_n)$  to gain information about the group of rational points on  $E_n$  is referred to as “full 2-descent”. Since the rank of  $S^{(2)}(E_n)$  is more accessible than  $r(n)$ , it has attracted great interest in recent years (for example see [3, 4, 5, 6, 7, 9]).

A second approach via descent to estimating  $r(n)$  is the method of descent via isogeny. In this approach, the sizes of  $S^{(\phi)}(E_n)$  and  $S^{(\hat{\phi})}(E_n)$  are used to approximate  $r(n)$ . In this approach the fundamental inequality that relates  $S^{(\phi)}(E_n)$  and  $S^{(\hat{\phi})}(E_n)$  to  $r(n)$  is

$$(1.3) \quad r(n) \leq s^\phi(n) + s^{\hat{\phi}}(n).$$

We count the number of square-free integers up to  $X$  that have 2-Selmer group of a given size. This problem was taken up and answered precisely by Heath-Brown in a pair of papers [6, 7]. Let  $\lambda := \prod_{j=1}^{\infty} (1 + 2^{-j})^{-1}$ , and for  $r = 0, 1, 2, \dots$  let  $d_r := \lambda \frac{2^r}{\prod_{1 \leq j \leq r} (2^j - 1)}$  and let

$$S(X, h) := \{1 \leq n \leq X : n \equiv h \pmod{8}, n \text{ square-free}\}.$$

If  $h = 1$  or  $3$  and  $r$  is even, or if  $h = 5$  or  $7$  and  $r$  is odd, then he showed that

$$(1.4) \quad |\{n \in S(X, h) : s(n) = r\}| \sim d_r |S(X, h)|.$$

We use different techniques and obtain related results. Heath-Brown remarks on page 336 of [7] that one should consider the rate of convergence to the limiting distribution in (1.4) as depending on the number of prime factors of the number  $n$ . For this reason we consider the problem of determining an asymptotic for  $n \in S(X, h, k)$  with 2-Selmer groups a given size, where  $S(X, h, k)$  is the set of all square-free integers less than  $X$  congruent to  $h$  modulo 8 which have exactly  $k$  prime factors.

In contrast to the theorems of [6, 7] that deal with full 2-descent, our theorems are concerned with the descent via isogeny.

**Theorem 1.1.** *In the notation above we have*

$$\begin{aligned} |\{n \in S(X, 3, k) : s^\phi(n) = s^{\hat{\phi}}(n) = 0\}| &= c_3(k)(1+o(1)) |S(X, 3, k)| \\ &= c_3(k)(1+o(1))X \cdot \frac{(\log \log X)^{k-1}}{4(k-1)! \log X}, \\ |\{n \in S(X, 2, k) : s^\phi(n) = s^{\hat{\phi}}(n) = 0\}| &= c_2(k)(1+o(1)) |S(X, 2, k)| \\ &= c_2(k)(1+o(1))X \cdot \frac{(\log \log X)^{k-2}}{2(k-2)! \log X}, \end{aligned}$$

where  $c_3(k) := \frac{k}{2^{k-1}} q(k)$ ,  $c_2(k) := \frac{2^{k-1}-1}{2^{2k-2}} q(k)$ , and  $q(k) := \prod_{j=1}^{\lfloor \frac{k}{2} \rfloor} \left(1 - \left(\frac{1}{2}\right)^{2j-1}\right)$ .

We also give a similar theorem for the Selmer groups  $S^{(\phi)}(E_n)$  and  $S^{(\hat{\phi})}(E_n)$ . Let  $\omega(n)$  be the number of prime factors of  $n$ .

**Theorem 1.2.** *Let  $R(X, k) := \{n < X : n \text{ square-free}, n \equiv 5 \pmod{8} \text{ with all prime factors } \equiv 1 \pmod{4}, \omega(n) = k\}$  and  $K(X, k) := \{n < X : \text{all prime factors } \equiv 1 \pmod{8}, \omega(n) = k, n \text{ square-free}\}$ . If  $0 \leq r \leq k-1$ , then*

$$\begin{aligned} |\{n \in R(X, k) : s^\phi(n) = r+1\}| &= q(k, r)(1+o(1)) |R(X, k)|, \\ |\{n \in R(X, k) : s^{\hat{\phi}}(n) = r+2\}| &= q(k, r)(1+o(1)) |R(X, k)|, \\ |\{n \in K(X, k) : s^\phi(n) = r+2\}| &= q(k, r)(1+o(1)) |K(X, k)|, \\ |\{n \in K(X, k) : s^{\hat{\phi}}(n) = r+2\}| &= q(k, r)(1+o(1)) |K(X, k)|, \end{aligned}$$

where

$$q(k, s) := d(k-1, s) \cdot 2^{\binom{k-s}{2} - \binom{k}{2}} \prod_{j=1}^{\lfloor \frac{k-s}{2} \rfloor} \left(1 - \left(\frac{1}{2}\right)^{2j-1}\right) \text{ and } d(m, s) = \prod_{i=0}^{r-1} \frac{2^m - 2^i}{2^s - 2^i}.$$

Additionally,  $|R(X, k)| = (1+o(1)) \frac{1}{2^{k+1}(k-1)!} X \cdot \frac{(\log \log X)^{k-1}}{\log X}$  and  $|K(X, k)| = (1+o(1)) \frac{1}{4^k(k-1)!} X \cdot \frac{(\log \log X)^{k-1}}{\log X}$ .

Conjecturally, the rank of an elliptic curve is related to analytic behavior of the  $L$ -function associated to  $E_n$ . Let  $L(E_n, s)$  be the  $L$ -function associated to the elliptic curve  $E_n$ . Then we have the following conjecture of Birch and Swinnerton-Dyer.

**Conjecture** (Birch and Swinnerton-Dyer for  $E_n$ ). *If  $E_n : y^2 = x^3 - n^2x$ , then  $\text{ord}_{s=1} L(E_n, s) = r(n)$ . Further if  $r(n) = 0$ , then*

$$(1.5) \quad L(E_n, 1)/\Omega_n = 2^{\ell(n)} |\text{III}(E_n)|,$$

where the constant  $\Omega_n$  is given by

$$(1.6) \quad \Omega_n := \frac{1}{\sqrt{n}} \int_1^\infty (x^3 - x)^{-1/2} dx$$

and  $\ell(n)$  is a non-negative integer. Also,  $\text{III}(E_n)$  is the Tate-Shafarevich group of  $E_n$  over  $\mathbb{Q}$ .

A special case of a famous theorem of Rubin [15, 16], implies that since  $E_n$  has complex multiplication by  $\mathbb{Z}[i]$ , if  $L(E_n, 1) \neq 0$ , then the group  $\text{III}(E_n/\mathbb{Q})$  is finite and the odd parts of both sides of equation (1.5) are equal. Thus, Rubin's result proves the Birch and Swinnerton-Dyer conjecture for  $E_n$  with  $r(n) = 0$  up to a power of 2. For many cases we are able to compute the power of 2 appearing in both sides of equation (1.5) and show that they are equal. Hence, we conclude the truth of the Birch and Swinnerton-Dyer conjecture for the elliptic curve  $E_n$ , for many values of  $n$ .

An advantage of the approach used here, that is restricting the number of prime factors, is that we can analyze the analytic properties of the  $L$ -functions at the same time we study the arithmetic properties of  $E_n$ . This is a result of the fact that the same combinatorial

conditions used to analyze the size of the 2-Selmer groups appear in the analysis of the 2-power in the  $L$ -value. In fact, Zhao in a series of papers [19, 20, 21, 22] described these conditions. Combining our results with Zhao's work and the work of Feng and Xiong [5], gives the following:

**Theorem 1.3.** *Let  $q(k)$  be as in Theorem 1.1.*

- (1) *Let  $B_3(X, k)$  be the set of all  $n < X$  with  $\omega(n) = k$ ,  $n \equiv 3 \pmod{8}$ , where  $n$  has exactly one prime factor congruent to 3 modulo 8, all other prime factors are 1 modulo 8. For any  $k$ , the Birch and Swinnerton-Dyer Conjecture is true for all  $n \in B_3(\infty, k)$  with  $s^\phi(n) = s^{\hat{\phi}}(n) = 0$ . Moreover, we have that*

$$\left| \{n \in B_3(X, k) : s^\phi(n) = s^{\hat{\phi}}(n) = 0\} \right| = q(k)(1 + o(1)) |B_3(X, k)|.$$

*Additionally,  $|B_3(X, k)| = (1 + o(1)) \frac{k}{4^{k(k-1)!}} X \cdot \frac{(\log \log X)^{k-1}}{\log X}$ .*

- (2) *Let  $B_2(X, k)$  be the set of all  $n < X$  with  $n \equiv 2 \pmod{8}$ ,  $\omega(n/2) = k$ ,  $n$  has all odd prime factors congruent to 1 modulo 4. For any  $k$ , the Birch and Swinnerton-Dyer Conjecture is true for all  $n \in B_2(\infty, k)$  with  $s^\phi(n) = s^{\hat{\phi}}(n) = 0$ . Moreover, we have that*

$$\left| \{n \in B_2(X, k) : s^\phi(n) = s^{\hat{\phi}}(n) = 0\} \right| = \frac{2^k - 1}{2^k} q(k)(1 + o(1)) |B_2(X, k)|.$$

*Additionally,  $|B_2(X, k)| = (1 + o(1)) \frac{1}{2^{k+1}(k-1)!} X \cdot \frac{(\log \log X)^{k-1}}{\log X}$ .*

*Remark.* This theorem gives information about the number of twists of  $L(E_1, s)$  which have  $L(E_n, 1) \neq 0$ . Ono and Skinner [13, 14] have given much more general results which establish lower bounds for the number of twists of an  $L$ -function which have non-vanishing of the central value. Similar to this result, their results are based on showing the “oddness” of the algebraic part of the  $L$ -value.

Theorem 1.3 verifies BSD for all curves  $E_n$  with trivial 2-Selmer groups  $S^{(\phi)}(E_n)$  and  $S^{(\hat{\phi})}(E_n)$  such that the prime factors of  $n$  are subject to some congruence conditions. It is possible to use the work of Zhao [22] and Li and Tian [11] to verify the full BSD conjecture for an infinite class of curves  $E_n$  whose Tate-Shafarevich group has non-trivial 2-part.

**Theorem 1.4.** *Let  $D_1(X, k)$  be the set of all  $n < X$  with  $\omega(n) = k$ ,  $n \equiv 1 \pmod{8}$ ,  $n$  has all prime factors congruent to 1 modulo 8. Then for any  $k$ , the Birch and Swinnerton-Dyer Conjecture is true for all  $n \in D_1(\infty, k)$  with  $s^\phi(n) = 2$ ,  $s^{\hat{\phi}}(n) = 0$  and  $\text{III}(E_n)[2] = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . Also if the conditions on the Selmer groups and Tate-Shafarevich group are satisfied then  $r(n) = 0$ . Finally,*

$$\begin{aligned} & \left| \{n \in D_1(X, k) : s^\phi(n) = 2, s^{\hat{\phi}}(n) = 0, \text{III}(E_n)[2] = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\} \right| \\ & \geq q(k) \frac{1}{2} (1 + o(1)) |D_1(X, k)| = \frac{q(k)}{2^{2k+1}(k-1)!} (1 + o(1)) X \cdot \frac{(\log \log X)^{k-1}}{\log X}. \end{aligned}$$

Briefly our approach to showing the vanishing of the  $\phi$  and  $\hat{\phi}$ -Selmer groups for the curve  $E_n$  follows from congruence conditions on the prime factors of  $n$  along with the equidistribution of the Legendre symbol. Feng [4] introduced the language of “odd graphs” to encode the necessary information about the Legendre symbol. In Section 2 we introduce his language and some related results that we will need in our analysis. Section 3 puts congruence conditions on a classical theorem of Landau which gives an asymptotic for the number of integers less than  $X$  with a fixed number of prime factors as  $X$  tends to infinity. The equidistribution of the Legendre symbol that is needed essentially follows from work of Cremona and Odoni [2] and is recalled in Section 4. In Section 5 we combine the results of Sections 2-4 to give the proofs of Theorems 1.1 and 1.2. Finally, in Section 6 we use Zhao’s results [19, 20, 21, 22] to verify some cases of the Birch and Swinnerton-Dyer Conjecture.

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#### 2. COUNTING SELMER GROUPS VIA ODD GRAPHS

The theory of “odd graphs”, initiated by Feng [4], has been used in many places to count Selmer groups, see [3, 4, 5]. This section describes two theorems of Feng and Xiong [5] that gives necessary and sufficient conditions for the triviality of the  $\phi$  and  $\hat{\phi}$ -Selmer groups. We will also describe some results of Faulkner and James [3] which we will use to prove Theorem 1.2. We now describe the graphs we will be interested in.

Throughout this section, unless otherwise stated, suppose that  $n$  is an odd square-free integer with  $n = p_1 \dots p_t > 0$ . Define the directed graphs  $G(n)$ ,  $G(-n)$ ,  $G'(n)$  by

$$(2.1) \quad V(G(n)) := \{p_1, \dots, p_t\} \quad \text{and} \quad E(G(n)) := \left\{ \overrightarrow{p_i p_j} : \left( \frac{p_i}{p_j} \right) = -1, 1 \leq i \neq j \leq t \right\},$$

$$(2.2) \quad V(G(-n)) := \{-1, p_1, \dots, p_t\} \quad \text{and}$$

$$(2.3) \quad E(G(-n)) := \left\{ \overrightarrow{p_i p_j} : \left( \frac{p_i}{p_j} \right) = -1, 1 \leq i \neq j \leq t, p_i \not\equiv 3 \pmod{4} \right\} \\ \cup \left\{ \overrightarrow{(-1)r} : r \in \{p_1, \dots, p_t\}, r \equiv \pm 3 \pmod{8} \right\},$$

$$(2.4) \quad V(G'(n)) := \{2, p_1, \dots, p_t\} \quad \text{and}$$

$$(2.5) \quad E(G'(n)) := \left\{ \overrightarrow{p_i p_j} : \left( \frac{p_i}{p_j} \right) = -1, 1 \leq i \neq j \leq t, p_i \not\equiv 3 \pmod{4} \right\} \\ \cup \left\{ \overrightarrow{r^2} : r \in \{p_1, \dots, p_t\}, r \equiv \pm 5 \pmod{8} \right\},$$

where  $V(\cdot)$  and  $E(\cdot)$  stand for the vertex set and edge sets of the graph.

**Definition 2.1.** Suppose that  $G$  is a graph with vertex set  $V$  and edge set  $E$ . A *partition* of  $G$  is a pair  $(S, T)$  of sets such that  $S \cap T = \emptyset$  and  $S \cup T = V$ . A partition  $(S, T)$  is *even* provided that all  $v \in S$  have an even number of edges directed from  $v$  to vertices in  $T$  and all  $v \in T$  have an even number of edges directed from  $v$  to vertices in  $S$ .

In particular, the partitions  $(G, \emptyset)$  and  $(\emptyset, G)$  are always even partitions. We call these the trivial partitions. Let  $e(G)$  be the number of even partitions of the graph  $G$ .

**Definition 2.2.** A graph  $G$  is called *even* provided that it admits a nontrivial even partition. A graph  $G$  is said to be *odd* provided that its only even partitions are trivial.

We recall two theorems of [5] that we will use to obtain Theorem 1.1.

**Theorem 2.3** (Theorem 2.4 of [5]). *Suppose that  $n \equiv \pm 3 \pmod{8}$ . Then  $S^{(\phi)}(E_n) = \{1\}$  and  $S^{(\hat{\phi})}(E_n) = \{\pm 1, \pm n\}$  if and only if the following three conditions are satisfied:*

- (1)  $n \equiv 3 \pmod{8}$
- (2)  $n = p_1 \dots p_t$ ,  $p_1 \equiv 3 \pmod{4}$  and  $p_j \equiv 1 \pmod{4}$  for  $(2 \leq j \leq t)$ .
- (3)  $G(n)$  is an odd graph.

**Theorem 2.4** (Theorem 2.6 of [5]). *Suppose that  $2 \parallel n$  then  $S^{(\phi)}(E_n) = \{1\}$  and  $S^{(\hat{\phi})}(E_n) = \{\pm 1, \pm n\}$  if and only if  $G'(n/2)$  is odd. Furthermore, if  $S^{(\phi)}(E_n) = \{1\}$  and  $S^{(\hat{\phi})}(E_n) = \{\pm 1, \pm n\}$  then all odd primes dividing  $n$  are 1 modulo 4 and there is at least one that is 5 modulo 8.*

We now state the results of Faulkner and James that we will use. The following combines special cases of Theorems 1.4 and 1.5 of [3].

**Theorem 2.5.** *With the notation from above, we have*

- (1) *If  $n \equiv 5 \pmod{8}$ , and  $n$  has all primes congruent to 1 modulo 4, then*

$$|S^{(\phi)}(E_n)| = e(G(n)),$$

*and*

$$|S^{(\hat{\phi})}(E_n)| = 2 \cdot e(G(n)).$$

- (2) *If  $n \equiv 1 \pmod{8}$  and all the prime factors of  $n$  are congruent to 1 modulo 8, then*

$$|S^{(\phi)}(E_n)| = 2 \cdot e(G(n)),$$

*and*

$$|S^{(\hat{\phi})}(E_n)| = e(G(-n)).$$

**2.1. Approach For Asymptotics.** The approach for using Theorems 2.3 and 2.4 to prove Theorem 1.1 is the following: For any  $n$  that satisfies the first two conditions of Theorem 2.3, by quadratic reciprocity  $G(n)$  is an undirected graph. Furthermore, by Dirichlet's theorem on primes in arithmetic progressions and induction on the number of prime factors of  $n$ , we see that for any undirected graph  $G$  there exist infinitely many  $n$  such that  $G(n) = G$ .

Given that there are infinitely many  $n$  such that each undirected graph  $G$  appears as  $G(n)$ , one might hope that selecting  $n$  (that satisfies the first two conditions of Theorem 2.3) at random would result in selecting a random graph  $G(n)$ . To make this more precise

we fix an integer  $k$ . Say  $\{n_1, n_2, n_3, \dots, n_M\}$  is the set of all appropriate integers less than some  $X$  with  $k$  prime factors. Then we might hope that if we look at the list of graphs  $(G(n_1), G(n_2), \dots, G(n_M))$  then each undirected graph on  $k$  vertices appears in the list with equal proportion. If this is true, then the proportion of  $n \in \{n_1, n_2, \dots, n_M\}$  that have  $S^{(\phi)}(n)$  and  $S^{(\hat{\phi})}(n)$  trivial should be the same as the probability that a random undirected graph on  $k$  vertices is odd.

**2.2. Probability of Odd Graphs.** Before moving on, we recall the results of [1] about the probability that a graph on  $k$  vertices is odd and some results connecting the number of odd partitions of a graph and the rank of an associated matrix over  $\mathbb{F}_2$ .

If  $G = (V, E)$  is a graph with vertices  $v_1, \dots, v_k$ , then define the adjacency matrix  $A(G)$  of a graph  $G$  by  $A(G) = (a_{ij})_{1 \leq i, j \leq k}$  where for  $i \neq j$ ,  $a_{ij} = 1$  if  $\overrightarrow{v_i v_j} \in E(G)$  and 0 otherwise and  $a_{ii} = 0$ . Let  $d_i = \sum_{j=1}^k a_{ij} \pmod{2}$ . The Laplace matrix of  $G$  is defined by  $L(G) = \text{diag}(d_1, \dots, d_k) + A(G)$ .

**Lemma 2.6** (Lemma 2.2 [5]). *Let  $G = (V, E)$  be a directed graph,  $k = |V|$  and  $r = \text{rank}_{\mathbb{F}_2} L(G)$ . Then the number of even partitions of  $G$  is  $2^{k-r}$ . In particular,  $G$  is an odd graph if and only if  $r = k - 1$ .*

*Remark.* In fact, [3] shows an explicit relationship between elements of the kernel of the matrix  $L(G)$  and elements of the Selmer groups under consideration.

**Theorem 2.7** (Theorem 1.6 [1]). *Let  $G$  be an undirected graph on  $k$  vertices. Denote the probability that  $G$  has  $2^{e+1}$  even partitions by  $q(k, e)$ , for  $0 \leq e \leq k - 1$ . Then*

$$q(k, e) = 2^{\binom{k-e}{2} - \binom{k}{2}} d(k-1, e) \prod_{j=1}^{\lfloor \frac{k-e}{2} \rfloor} \left( 1 - \left( \frac{1}{2} \right)^{2j-1} \right),$$

where  $d(m, j) := \prod_{i=0}^{j-1} \frac{2^m - 2^i}{2^j - 2^i}$ .

Also in [1] one finds the following proposition.

**Proposition 2.8.** *Denote the probability that a  $k \times k$  matrix over  $\mathbb{F}_2$  has rank  $k$  by  $p(k)$ . Then  $p(k) = q(k+1, 0) = \prod_{j=1}^{\lfloor (k+1)/2 \rfloor} (1 - 2^{-2j+1})$ , where  $q(k+1, k)$  is as in Theorem 2.7.*

We will need the following proposition as well.

**Proposition 2.9.** *Let  $A$  be a  $k \times k$  symmetric matrix over  $\mathbb{F}_2$ . Given that the sum of the rows of  $A$  is  $v = (0, \dots, 0, 1, \dots, 1)^T$  for some vector with  $j > 0$  1's, the probability that  $A$  has rank  $k$  is  $p(k-1) = q(k, 0)$ , independent of  $j$ .*

*Proof.* If  $j = 1$ , then apply the remark after Theorem 2.6 of [5]. If  $j \geq 2$ , then we can find a change of basis matrix  $\Lambda$  such that  $\Lambda v = (0 \dots 0 1)^T$ . Thus  $\Lambda^T A \Lambda$  is a symmetric matrix with the same rank as  $A$  and we are reduced to the  $j = 1$  case.  $\square$

*Remark.* Monsky, in an appendix to [7], gives a way to compute the size of  $S^{(2)}(n)$  by computing the  $\mathbb{F}_2$  rank of larger matrix than any of the ones we consider here.

### 3. SQUARE-FREE INTEGERS WITH FACTORS IN SPECIFIED CONGRUENCE CLASSES

In this section we answer the question of how many integers  $n < X$  have exactly  $k$  prime factors where each lies in a specified congruence class modulo  $m$ , for some  $m$ . The most classical result in this direction, due to Landau, states:

$$(3.1) \quad |\{n \leq X : \Omega(n) = k\}| \sim |\{n \leq X : \omega(n) = k\}| \sim \frac{X(\log \log X)^{k-1}}{(k-1)! \log X},$$

where  $\Omega(n)$  is the number of prime factors of  $n$  counted with multiplicity and  $\omega(n)$  is the number of prime factors of  $n$  counted without multiplicity.

We fix the following notation: for a fixed positive integer  $m > 1$ , we let  $1 = r_1 < \dots < r_{\phi(m)} < m$  be the  $\phi(m)$  standard representatives for  $(\mathbb{Z}/m\mathbb{Z})^\times$ . Define  $\pi_k(X; m; a_1, \dots, a_{\phi(m)})$  to be the number of  $n \leq X$  with  $\omega(n) = k$  and  $n$  is square-free with exactly  $a_j$  prime factors congruent to  $r_j \pmod{m}$  for  $1 \leq j \leq \phi(m)$ . We have the following theorem.

**Theorem 3.1.** *Let  $k$  and  $m$  be fixed positive integers with  $m > 1$  and let  $X$  be a positive real number. If  $0 \leq a_1, a_2, \dots, a_{\phi(m)} \leq k$  are integers such that  $a_1 + a_2 + \dots + a_{\phi(m)} = k$ , then*

$$\pi_k(X; m; a_1, a_2, \dots, a_{\phi(m)}) = (1 + o(1)) \frac{k!}{a_1! a_2! \dots a_{\phi(m)}!} \frac{1}{\phi(m)^k (k-1)!} \frac{X(\log \log X)^{k-1}}{\log(X)}.$$

We will prove Theorem 3.1 by induction on the number of non-zero  $a_j$ . The base case is where just one of the  $a_j$  is non-zero. As short hand we write  $\pi_k(X; m; b) = \pi_k(X; m; a_1, \dots, a_{\phi(m)})$  where there is a  $j$  with  $r_j = b$  and  $a_j = k$ . In this case we necessarily have  $a_i = 0$  for all  $i \neq j$ .

**Proposition 3.2.** *If  $m$  is a positive integer greater than 1 and  $b$  is a positive integer relatively prime to  $m$ , then*

$$\pi_k(X; m; b) = (1 + o(1)) \frac{1}{\phi(m)^k (k-1)!} \frac{X(\log \log X)^{k-1}}{\log(X)}.$$

This result follows from a trivial modification of the proof of Landau's result. For a proof of Landau's result see [12]. To complete the induction we will need the following two lemmas.

**Lemma 3.3.** *Let  $k$  be a positive integer, and  $C$  a positive real number. Let  $S$  be a set of positive integers such that*

$$S(X) := |\{n \leq X : n \in S\}| = (1 + o(1)) \frac{C}{(k-1)!} \frac{X(\log \log X)^{k-1}}{\log(X)},$$

as  $X \rightarrow \infty$ . We have that

$$\int_2^{X^{1/2}} \frac{S(t)}{t^2 \log(X/t)} dt = (1 + o(1)) \frac{C}{k!} \frac{(\log \log X)^k}{\log(X)}.$$

*Proof.* For any  $\epsilon > 0$ , there exists an  $N$  such that for all  $X > N$ ,

$$\left| S(X) - \frac{C}{(k-1)!} \frac{X(\log \log X)^{k-1}}{\log X} \right| < \epsilon \frac{C}{(k-1)!} \frac{X(\log \log X)^{k-1}}{\log X}.$$



Therefore it follows that

$$\begin{aligned} \int_2^{X^{1/2}} \frac{S(t)}{t^2 \log(X/t)} dt &= \int_2^N \frac{S(t)}{t^2 \log(X/t)} dt + \int_N^{X^{1/2}} \frac{S(t)}{t^2 \log(X/t)} dt \\ &= O\left(\frac{\log N}{\log X}\right) + \int_N^{X^{1/2}} \frac{S(t)}{t^2 \log(X/t)} dt. \end{aligned}$$

Where we estimate the first integral by using the fact that  $S$  is a set of positive integers and so  $S(t) \leq t$ .

Now we turn to estimating  $\int_N^{X^{1/2}} \frac{S(t)}{t^2 \log(X/t)} dt$ . In the range of integration we know that we may replace  $S(t)$  by its asymptotic formula and introduce a small error. Precisely we have

$$\left| \int_N^{X^{1/2}} \frac{S(t)}{t^2 \log(X/t)} dt - \frac{C}{(k-1)!} \int_N^{X^{1/2}} \frac{t(\log \log t)^{k-1}}{t^2 \log(t) \log(X/t)} dt \right| \leq \epsilon \frac{C}{(k-1)!} \int_N^{X^{1/2}} \frac{t(\log \log t)^{k-1}}{t^2 \log(t) \log(X/t)} dt.$$

We have

$$\begin{aligned} \int_N^{X^{1/2}} \frac{(\log \log t)^{k-1}}{t \log(t) \log(X/t)} dt &= \frac{1}{\log(X)} \int_N^{X^{1/2}} \frac{(\log \log t)^{k-1}}{t \log(t)} \left(1 + O\left(\frac{\log(t)}{\log(X)}\right)\right) dt \\ &= \frac{(\log \log X)^k}{k \log(X)} + O\left(\frac{1}{(\log X)^2} \int_N^{X^{1/2}} \frac{(\log \log t)^{k-1}}{t} dt\right) \\ &= \frac{(\log \log X)^k}{k \log(X)} + O\left(\frac{(\log \log X)^{k-1}}{\log X}\right). \end{aligned}$$

Thus we may conclude that

$$\int_2^{X^{1/2}} \frac{S(t)}{t^2 \log(X/t)} dt = \frac{C(\log \log X)^k}{k! \log(X)} (1 + o(1)) + O\left(\frac{\log N}{\log X}\right).$$

Taking  $N$  to be of size  $\log \log X$  is sufficient to yield the conclusion of the lemma.  $\square$

**Lemma 3.4.** *Let  $k_1, k_2$  be positive integers and  $C_1$  and  $C_2$  be positive numbers. Let  $S_1$  and  $S_2$  be two sets of positive integers such that for each  $j$*

$$S_j(X) := |\{n \leq X : n \in S_j\}| = (1 + o(1)) \frac{C_j}{(k_j - 1)!} \frac{X(\log \log X)^{k_j - 1}}{\log(X)},$$

as  $X \rightarrow \infty$ . We have that

$$S_{1,2}(X) := |\{(n_1, n_2) \in S_1 \times S_2 : n_1 n_2 \leq X\}| = (1 + o(1)) \frac{C}{(k_1 + k_2 - 1)!} \frac{X(\log \log X)^{k_1 + k_2 - 1}}{\log X},$$

where  $C = \frac{C_1 C_2 (k_1 + k_2 - 1)!}{(k_1 - 1)! (k_2 - 1)!} \left(\frac{1}{k_1} + \frac{1}{k_2}\right)$ .

*Proof.* We abuse notation and refer to  $S_j(X)$  as the set of elements in  $S_j$  which are less than or equal to  $X$ , as well as the size of the set of elements of  $S_j$  up to  $X$ . The use of the symbol will be clear from the context.

Begin with the following inclusion-exclusion-like identity

$$S_{1,2}(X) = \sum_{t \in S_2(X^{1/2})} S_1(X/t) + \sum_{t \in S_1(X^{1/2})} S_2(X/t) - S_1(X^{1/2})S_2(X^{1/2}).$$

This identity follows from the fact that the first two sums will count everything in the set  $S_{1,2}(X)$ , while over counting precisely those elements which equal  $n_1 n_2$  where  $n_1 \in S_1$  and  $n_2 \in S_2$  and both  $n_1$  and  $n_2$  are smaller than  $X^{1/2}$ .

By assumption we have

$$S_1(X^{1/2})S_2(X^{1/2}) = O\left(\frac{X(\log \log X)^{k_1+k_2-2}}{(\log X)^2}\right),$$

and this is well within our expected error. We now estimate the first sum.

Since  $t \leq X^{1/2}$  we know that  $X/t \geq X^{1/2}$ , so we may apply our asymptotic to obtain

$$\sum_{t \in S_2(X^{1/2})} S_1(X/t) = (1 + o(1)) \frac{C_1}{(k_1 - 1)!} X \sum_{t \in S_2(X^{1/2})} \frac{(\log \log X/t)^{k_1-1}}{t \log(X/t)}.$$

Using the fact that  $\log \log(X/t) = \log \log X + \log\left(1 - \frac{\log t}{\log X}\right) = \log \log X + O(1)$  for  $t \in [1, X^{1/2}]$ , we obtain

$$(3.2) \quad \sum_{t \in S_2(X^{1/2})} S_1(X/t) = (1 + o(1)) \frac{C_1}{(k_1 - 1)!} X (\log \log X)^{k_1-1} \sum_{t \in S_2(X^{1/2})} \frac{1}{t \log(X/t)}.$$

We have

$$\begin{aligned} \sum_{t \in S_2(X^{1/2})} \frac{1}{t \log(X/t)} &= \int_2^{X^{1/2}} \frac{1}{t \log(X/t)} dS_2(t) \\ &= \frac{S_2(X^{1/2})}{X^{1/2} \log(X^{1/2})} - \int_2^{X^{1/2}} \frac{S_2(t)}{t^2 (\log(X/t))^2} dt + \int_2^{X^{1/2}} \frac{S_2(t)}{t^2 \log(X/t)} dt \\ &= O\left(\frac{(\log \log X)^{k_2-1}}{(\log X)^2}\right) - O\left(\frac{(\log \log X)^{k_2}}{(\log X)^2}\right) + (1 + o(1)) \frac{C_2 (\log \log X)^{k_2}}{k_2! \log(X)} \\ &= (1 + o(1)) \frac{C_2 (\log \log X)^{k_2}}{k_2! \log(X)}, \end{aligned}$$

where we use the fact that

$$\int_2^{X^{1/2}} \frac{S_2(t)}{t^2 (\log(X/t))^2} dt \ll \frac{1}{\log X} \int_2^{X^{1/2}} \frac{S_2(t)}{t^2 \log(X/t)} dt$$

and Lemma 3.3 to estimate the first integral.

Combining this with equation (3.2) we obtain

$$\sum_{t \in S_2(X^{1/2})} S_1(X/t) = (1 + o(1)) \frac{C_1 C_2}{(k_1 - 1)! k_2!} \frac{X (\log \log X)^{k_1+k_2-1}}{\log X}.$$

The exact same argument for the sum  $\sum_{t \in S_1(X^{1/2})} S_2(X/t)$ , shows that

$$S_{1,2}(X) = (1 + o(1)) \left( \frac{C_1 C_2}{(k_1 - 1)! k_2!} + \frac{C_1 C_2}{k_1! (k_2 - 1)!} \right) \frac{X (\log \log X)^{k_1 + k_2 - 1}}{\log X}.$$

□

With this lemma in hand we prove Theorem 3.1.

*Proof of Theorem 3.1.* Using Proposition 3.2 as the base case, the result now follows by induction with a straightforward application of Lemma 3.4. □

#### 4. INDEPENDENCE OF LEGENDRE SYMBOLS

The main theorem of this section is to prove that the Legendre symbols are independent; this allows us to conclude that the graphs discussed in Section 2 are asymptotically uniformly distributed. This theorem is a simple extension of the results in Section 3 of [2]. As a result we do not include the proof here. Instead we only remark that to obtain this theorem one would follow the argument of [2] but would need to add the additional constraint on the  $L$ -functions considered to take into account the additional congruence conditions on the prime factors  $p_j$ . Specifically, we have the following theorem:

**Theorem 4.1.** *Let  $k \geq 2$  be a positive integer. Fix  $\epsilon_{ij} \in \{-1, 1\}$  and  $\delta_j \in \{1, 3, 5, 7\}$  for  $1 \leq j \leq k$  and  $1 \leq i < j \leq k$ . For ease of notation let  $\delta = (\delta_1, \dots, \delta_k)$ . Let  $C_k(X, \delta)$  be the set of  $k$ -tuples  $(p_1, \dots, p_k)$  of primes with  $2 < p_1 < p_2 < \dots < p_k \leq X$ ,  $p_1 \cdots p_k \leq X$ ,  $p_j \equiv \delta_j \pmod{8}$ . Then the number of elements of  $C_k(X, \delta)$  with  $\left(\frac{p_i}{p_j}\right) = \epsilon_{ij}$  for  $i < j$ , is*

$$2^{-\binom{k}{2}} (1 + o(1)) |C_k(X, \delta)|.$$

#### 5. SELMER GROUP ASYMPTOTICS

In this section we give the proofs of Theorems 1.1 and 1.2. The strategy for the proofs was explained in Section 2. Here we quickly give the proofs, which amount to combining the results from the previous sections.

*Proof of Theorem 1.1.* Begin with the case  $n = p_1 p_2 \cdots p_k \equiv 3 \pmod{8}$ . Then by Theorem 2.3 and Lemma 2.6, we know that  $s^\phi(n) = s^{\hat{\phi}}(n) = 0$  if and only if  $n = p_1 p_2 \cdots p_k$  with  $p_1 \equiv 3 \pmod{4}$ ,  $p_j \equiv 1 \pmod{4}$ , and  $G(n)$  has exactly 2 even partitions (after suitable renaming the  $p_j$ ). By Theorem 3.1, we know that the number of  $n \equiv 3 \pmod{8}$  with the necessary congruence conditions on the prime factors is  $\frac{k}{2^{k-1}} (1 + o(1)) |S(X, 3, k)|$ . By Theorems 2.7 and 4.1, we know that of all the  $n$  with the necessary congruence conditions on the prime factors the proportion of them with  $G(n)$  odd is  $q(k, 0)(1 + o(1)) = q(k)(1 + o(1))$ . The result follows.

The proof for  $n \equiv 2 \pmod{8}$  is similar. However instead of appealing to Theorem 2.7 we use Proposition 2.9. From Theorem 2.4, we know that if  $s^\phi(n) = s^{\hat{\phi}}(n) = 0$  for some

$n \equiv 2 \pmod{8}$ , then there are no primes congruent to 3 modulo 4 that divide  $n$ . Say  $n = 2p_1 \cdots p_{k-1}$  has  $k - 1$  odd prime factors. Hence  $L(G'(n))$  is a  $k \times k$  matrix with

$$L(G'(n)) = \begin{pmatrix} A & v \\ 0 \cdots 0 & 0 \end{pmatrix},$$

where  $A$  is a symmetric  $(k - 1) \times (k - 1)$  matrix determined by  $\left(\frac{p_i}{p_j}\right)$  and the vector  $v$  is a matrix with the same number of 1's as primes congruent to 5 modulo 8, which by Theorem 2.4 is necessarily larger than zero. Now we know that the sum of the rows of  $L(G'(n))$  is 0. So we know, by Lemma 2.6 that the graph  $G'(n)$  is odd if and only if the matrix  $A$  has full rank. We apply Proposition 2.9 to see the probability that  $G'(n)$  is odd is  $q(k - 1, 0) = q(k - 1)$ . Finally, we apply Theorem 4.1 to justify that each possible  $k \times k$  matrix appears with equal probability.  $\square$

*Proof of Theorem 1.2.* Our starting point is Theorem 2.5. This proof is similar to the proof of Theorem 1.1, however it is important to realize that because the conditions on the prime factors of  $n$  the graphs are all undirected or equivalently all the matrices  $L(G(n))$  or  $L(G(-n))$  are symmetric matrices. Therefore we may apply Theorem 2.7 to determine the probability that the matrix has a given rank. We apply Theorem 4.1 to see that it is appropriate to treat the graphs appearing for such  $n$  as random undirected graphs.  $\square$

## 6. VERIFYING BSD

In this section we prove Theorems 1.3 and 1.4. The proof of the first of these theorems amounts to combining results from the work of Feng and Xiong, [5], the work of Zhao [19, 21, 22], and the results from this paper. The second theorem uses some work of Zhao [20] and work of Li and Tian [11].

Because the work of Zhao is important we will state one of his three theorems which we will employ. Let  $L(\overline{\psi}_{n^2}, s)$  denote the Hecke  $L$ -function corresponding to the dual of  $\psi_{n^2}$  which is the Grössencharacter of  $\mathbb{Q}[i]$  attached to  $E_n$ . See [19] for more details. Let  $\Omega_n$  be as in equation (1.6).

**Theorem 6.1** (Theorem 2 of [19]). *Suppose  $n = p_1 \cdots p_m$  with  $p_1 \equiv 3 \pmod{8}$  and  $p_j \equiv 1 \pmod{8}$  for all  $j > 1$ . The power of 2 in  $L(\overline{\psi}_{n^2}, 1)/\Omega_n$  is greater than or equal to  $2m - 1$  with equality if and only if  $G(n)$  is odd.*

Recall that for  $n$  with the prime factorization of this theorem we know from Theorem 2.3 that  $s(n) = 0$  and thus  $\text{III}(E_n)$  is odd when  $G(n)$  is odd. Since the condition for the lowest power of 2 is the same here as it is for  $\text{III}(E_n)$  to be odd we are able to verify the Birch and Swinnerton-Dyer Conjecture. Indeed, Zhao gives:

**Proposition 6.2** (Proposition 3 of [19]). *Suppose  $n \equiv 3 \pmod{8}$ ,  $n$  has one prime factor congruent to 3 modulo 8 and all others congruent to 1 modulo 8. If  $G(n)$  is odd, then the Birch and Swinnerton-Dyer Conjecture is true.*

*Proof of Theorem 1.3.* Theorem 2.3 shows that  $n \in B_3(X, k)$ ,  $s^\phi(n) = s^{\hat{\phi}}(n) = 0$  if and only if the graph  $G(n)$  is odd. Proposition 6.2 shows that for such  $n$  the Birch and Swinnerton-Dyer conjecture is true. Finally, applying Theorem 2.7 and Theorem 4.1, we obtain the

asymptotic, as in the proof of Theorem 1.1. To prove the second case of this theorem, we use Theorem 2.4 and Corollary 3 of [21].  $\square$

The same proof using the main theorem of [22] with Theorem 2.5 of [5] gives the following proposition.

**Proposition 6.3.**

*Let  $B_1(X, k)$  be the set of all  $n < X$  with  $\omega(n) = k$ ,  $n \equiv 1 \pmod{8}$ ,  $n$  has exactly two prime factors congruent to 3 modulo 8, all other prime factors are 1 modulo 8. Then for any  $k$ , the Birch and Swinnerton-Dyer Conjecture is true for all  $n \in B_1(\infty, k)$  with  $s^\phi(n) = s^{\hat{\phi}}(n) = 0$ .*

Since the combinatorics of this case are a bit messier we do not bother to present the asymptotics. Before we can prove the final result of this paper, we must give one more definition and one more lemma.

**Definition 6.4.** Let  $p \equiv 1 \pmod{8}$  be prime. Then set  $\delta(p) = 1$  if we have one of the following:

- $p \equiv 1 \pmod{16}$  and  $\left(\frac{2}{p}\right)_4 = -1$
- $p \equiv 9 \pmod{16}$  and  $\left(\frac{2}{p}\right)_4 = 1$ ,

and set  $\delta(p) = 0$  otherwise. Here  $\left(\frac{2}{p}\right)_4$  is the quartic character. For an integer  $n = p_1 \cdots p_k$  with each  $p_j \equiv 1 \pmod{8}$ , set  $\delta(n) = \sum_{i=1}^k \delta(p_i) \pmod{2}$ .

The following result is important for Theorem 1.4

**Proposition 6.5** (Li and Tian [11]). *Let  $n \in D_1(X, k)$ , where  $D_1$  is as in Theorem 1.4. We have  $G(n)$  is an odd graph and  $\delta(n) = 1$  then  $s^\phi(n) = 2$ ,  $s^{\hat{\phi}}(n) = 0$ , and  $\text{III}(E_n)[2] = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .*

The converse direction should also be true, but for brevity we do not give the proof. Proving the converse direction would result in an equality in Theorem 1.4 rather than the lower bound given.

**Theorem 6.6** (Theorem 1 of [20]). *Suppose  $n \in D_1(X, k)$ . Then the power of 2 in  $L(\overline{\psi}_{n^2}, 1)/\Omega_n \geq 2k$  and there is equality if and only if  $\delta(n)$  is odd and  $G(n)$  is an odd graph.*

*Proof of Theorem 1.4.* For  $n \in D_1(X, k)$  we proceed as in the proof of Theorem 1.3. Zhao [22] proved that the power of 2 in the  $L$ -value is as small as possible if and only if  $G(n)$  is odd and  $\delta(n)$  is odd. The previous proposition proves that the power of 2 in  $\text{III}(E_n)$  is as small as possible (in this case 2) if and only if  $\delta(n)$  is odd and  $G(n)$  is odd. Now we obtain the asymptotic by applying Theorem 4.1 combined with Theorem 2.7 and noting that for  $|\{n \in D_1(X, k) : \delta(n) = 1\}| = (1/2 + o(1)) |D_1(X, k)|$ . Technically we would need a version of Theorem 4.1, which has  $\delta_j \in \{1, 9\}$  and we consider the  $p_j$  modulo 16 instead of 8. But the proof goes through the same as the case we handle.  $\square$

## REFERENCES

- [1] M. Brown, N. Calkin, A. King, K. James, S. Lockard, R.C. Rhoades, *Trivial Selmer Groups and Even Partitions*, Integers, November 2006.
- [2] J. Cremona and Odoni, *Some Density Results for Negative Pell Equations: an application of graph theory*, JLSM (1989) 16 - 28.
- [3] B. Faulkner and K. James *A graphical approach to computing Selmer groups of congruent number curves*, (preprint).
- [4] K. Feng, *Non-congruent numbers, odd graphs and the Birch-Swinnerton-Dyer conjecture*, *Acta Arithmetica*, (1996) 71 - 83.
- [5] K. Feng and M. Xiong, *On Elliptic Curves  $y^2 = x^3 - n^2x$  with Rank Zero*, Journal of Number Theory, Vol. 109, Issue 1, Nov. 2004, 1-26.
- [6] D. R. Heath-Brown, *The size of Selmer groups for the congruent number problem*, *Invent. Math.* **111** (1993), no.1, 171-195.
- [7] D.R. Heath-Brown, *The size of Selmer groups for the congruent number problem. II*. *Invent. Math.* 118 (1994), no. 2, 331-370
- [8] B. Iskra, *Non-congruent numbers with arbitrarily many prime factors congruent to 3 modulo 8*, Proc. Japan Acad. Ser. A Math. Sci. **72** (1996) no. 7, 168 - 169.
- [9] K. James, K. Ono, *Selmer groups of quadratic twists of elliptic curves*. Math. Ann. 314 (1999), no. 1, 1-17.
- [10] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer-Verlag, 1984.
- [11] D. Li and T. Tian, *On the Birch-Swinnerton-Dyer Conjecture of Elliptic Curves  $E_D : y^2 = x^3 - D^2x$* , *Acta Math. Sinica*, 2000 Vol. 16, No. 2, pp. 229 - 236.
- [12] M. Nathanson, *Elementary Methods in Number Theory*, Springer-Verlag, New York, 2000.
- [13] K. Ono, *Nonvanishing of quadratic twists of modular  $L$ -functions with applications for elliptic curves*, Journal für die reine und angewandte Mathematik **533**, 2001, pp. 81-97
- [14] K. Ono and C. Skinner, *Nonvanishing of quadratic twists of modular  $L$ -functions*, *Inventiones Mathematicae* **134**, 1998, pp. 651-660.
- [15] K. Rubin, *Tate-Shafarevich group and  $L$ -functions of elliptic curves with complex multiplication*, *Invent. Math.* **89** (1987), 527 - 560.
- [16] K. Rubin, *The main conjecture for imaginary quadratic fields*, *ibid.* **103** (1991), 25 - 68.
- [17] G. Yu, *Average size of 2-Selmer groups of elliptic curves, II*, *Acta Arith.* 117 (2005), no. 1, 1-33
- [18] G. Yu, *Average size of 2-Selmer groups of elliptic curves, I*, *Trans. AMS.* 358 (2005), no. 4, 1563-1584.
- [19] C. Zhao, *A criterion for elliptic curves with lowest 2-power in  $L(1)$* , *Math. Proc. Camb. Phil. Soc.* (1997), **121**, 385 - 400.
- [20] C. Zhao, *A criterion for elliptic curves with second lowest 2-power in  $L(1)$* , *Math. Proc. Camb. Phil. Soc.* (2001) **131** 385 - 400.
- [21] C. Zhao, *A criterion for elliptic curves with lowest 2-power in  $L(1)$ , II*, *Math. Proc. Cambridge Phil. Soc.* **134** (2003), no.3, 407-420.
- [22] C. Zhao, *A criterion for elliptic curves with second lowest 2-power in  $L(1)$ , II*, *Acta Math. Sin. (Engl. Ser.)* 21 (2005), no.5, 961- 976.

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